ALGEBRAIC GEOMETRY–BACK PAPER EXAM
B. MATH. III
TIME : 3 HOURS, MARKS : 80

All rings are assumed to be commutative with identity. All varieties are defined over an algebraically closed field k, unless specified otherwise.

You are allowed to bring two pages of hand written notes.

(1) (a) Let A be a ring such that for every element $x \in A$ there is some $n \in \mathbb{N}$ such that $x^n = x$. Show that every prime ideal of A is a maximal ideal.
(b) Show that the set of prime ideals of A contains minimal elements. [4+6]

(2) (a) If B is a flat A-algebra and N is a flat B-module, then N is flat as an A-module.
(b) Let $\phi : A^m \to A^n$ be a surjective A-linear map. Then $m \geq n$. [5+5]

(3) (a) Prove that localization preserves exact sequences.
(b) Let $f : M \to N$ be a map of A-modules. Then f is an isomorphism of A-modules iff for any $\mathfrak{p} \in \text{Spec}(A)$, the map $f_{\mathfrak{p}}$ is an isomorphism of $A_{\mathfrak{p}}$-modules. [6+4]

(4) (a) Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ iff there exists $s \in S$ such that $sM = 0$.
(b) If M is finitely generated, then $\text{Supp}(M) = \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \supseteq \text{Ann}(M)\}$. [6+4]

(5) Let A be a Noetherian ring. Prove that $A[X]$ is a Noetherian ring. [10]

(6) Let A be a subring of B such that B is integral over A, and let $f : A \to \Omega$ be a homomorphism of A into an algebraically closed field Ω. Show that f can be extended to a homomorphism of B into Ω. [10]

(7) Let A be a Noetherian ring.
(a) Show that the set of zero divisors of A can be written as a union of prime ideals of A.
(b) Let $\mathfrak{m}$ be a maximal ideal of A, $\mathfrak{q}$ be any ideal of A. Then the following are equivalent :
(i) $\mathfrak{q}$ is $\mathfrak{m}$-primary;
(ii) $\sqrt{\mathfrak{q}} = \mathfrak{m}$;
(iii) $\mathfrak{m}^n \subseteq \mathfrak{q} \subseteq \mathfrak{m}$ for some $n \in \mathbb{N}$. [6+4]

(8) Let $A \subset B$ be domains, B integral over A and A integrally closed. Let $\mathfrak{q}$ be a prime ideal of B and $\mathfrak{p} = \mathfrak{q} \cap A$. Then $\dim A_{\mathfrak{p}} = \dim B_{\mathfrak{q}}$. [10]

(9) Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic variety in $\mathbb{A}^n$, for some n, iff B is a finitely generated k-algebra which is a domain. [10]

(10) Let the characteristic of the base field k be $p > 0$. Define a map $\phi : \mathbb{A}^1 \to \mathbb{A}^1$ by $t \mapsto t^p$. Show that ϕ is a morphism which is a homeomorphism but not an isomorphism. [10]

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